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A METRIC CHARACTERIZATION OF CELLS

BY

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ABSTRACT. We examine finite dimensional compact convex metric spaces each having the property that the union of two line segments in the space, having more than one point in common, is a line segment. The question has been asked (Borsuk; Bing) whether each such space is a cell. The answer is yes if the dimension of the space is ≤ 2 (Lelek and Nitka) or 3 (Rolfsen). Here we provide an affirmative answer for arbitrary finite dimension provided the space has the additional property that the join of any point to any line segment in the space is a convex set.

I. α-metrics. A metric d for a space X is called *convex* if whenever x and y are points of X, there exists some point $p \in X$ such that $d(x, p) = \frac{1}{2} d(x, y) = d(y, p)$. *Throughout this paper X will denote a nondegenerate compact metric space with a convex metric d.* If p and q are distinct points of X, then there is in X an isometric image, $L(p, q)$, of the interval $[0, d(p, q)]$, having p and q as endpoints. The set $L(p, q)$ is called a line segment between p and q.

Definition 1. The metric d is called an α-metric if whenever $L(p, q)$ and $L(p', q')$ are line segments in X having more than one point in common, then $L(p, q) \cup L(p', q')$ is a line segment. (Note that if $L(p, q) \cup L(p', q') = L(s, t)$, then $\{s, t\} \subset \{p, q, p', q'\}$.)

Throughout the remainder of this section, d is assumed to be an α-metric.

There now follows several propositions and theorems concerning α-metrics, mostly without proofs. The proofs are straightforward, and many are similar in spirit to those found in [3], [4], and [7].

Proposition 1. *If p and q are distinct points of X, then $L(p, q)$ is unique.*

Corollary. *If p and q are distinct points of X, and $0 \leq r \leq 1$, then there is exactly one point $x \in X$ such that $d(p, x) = rd(p, q)$ and $d(q, x) = (1-r)d(p, q)$.*

As in [4] we define $\lambda: X \times X \times [0, 1] \to X$ as $\lambda(p, q, r) =$ the unique point $x \in X$ such that $d(p, x) = rd(p, q)$ and $d(q, x) = (1-r)d(p, q)$. It is easy to check that λ is continuous.

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Definition 2. A line segment $L(p, m)$ is called a maximal ray from p if whenever $L(p, m')$ is a segment such that $L(p, m) \subset L(p, m')$, then $L(p, m) = L(p, m')$, and hence $m = m'$. A line segment $L(a, b)$ is called a maximal segment if whenever $L(a', b')$ is a segment such that $L(a, b) \subset L(a', b')$, then $L(a, b) = L(a', b')$ and hence $\{a, b\} = \{a', b'\}$.

Theorem 1. *Suppose p and q are distinct points of X. Then* (1) *there exists a unique maximal ray from p, $L(p, m)$, such that $L(p, q) \subset L(p, m)$; and* (2) *there exists a unique maximal segment $L(n, m)$ containing $L(p, q)$.*

It follows from Proposition 1 and Theorem 1 that each pair of points of X is contained in a unique maximal line segment. In fact, if X is compact and d is convex, then d is an α-metric if and only if each pair of points of X is contained in a unique maximal line segment.

Definition 3. If $A \subset X$ and $p \in X$, then the p-face of A, denoted by $F_p(A)$, is defined as $F_p(A) = \{a \in A \mid L(p, a) \cap A = \{a\}\}$.

Proposition 2. *If $A \subset X$, $p \notin A$, $\{s, t\} \subset F_p(A)$, $s \neq t$, then $L(p, s) \cap L(p, t) = \{p\}$.*

Proof. Otherwise $L(p, s) \cup L(p, t) = L(p, s)$ (or $L(p, t)$). In which case $s \notin F_p(A)$ (or $t \notin F_p(A)$).

Proposition 3. *If L is a maximal line segment and $p \notin L$, then $L = F_p(L)$.*

If A is a closed, convex subset of X, then d is an α-metric for A. That is, a closed convex subset of an "α-space" is an "α-space".

Definition 4. Suppose A is a closed convex subset of X, By a *rim point* of A we mean a point of A which is an endpoint of some maximal segment in A. By the *rim* of A, denoted by $\partial(A)$, we mean the set of all rim points of A.

Theorem 2. *Suppose A is a closed subset of X, $p \in X - A$, and $A = F_p(A)$. Then $L(p, A) = \bigcup \{L(p, x) \mid x \in A\}$ is homeomorphic to the cone over A, $C(A)$.*

Proof. The obvious "linear" function from $C(A)$ to $L(p, A)$ is a homeomorphism.

In particular, if A is an n-cell, $n \geq 0$, then $L(p, A)$ is an $(n + 1)$-cell with $A \cup L(p, \text{bdry } A) = \text{bdry } L(p, A)$.

Proposition 4. *Suppose A is a closed, convex subset of X, $L(s, t)$ is a maximal line segment in X, and $L(s, t) \cap A$ is nondegenerate. Then $L(s, t) \cap A$ is a maximal line segment in A.*

Proposition 5. Suppose A is a closed, convex subset of X, $p \in X - A$, $q \in A - F_p(A)$. Then $L(p, q)$ contains a point of $\partial(A)$ in its interior.

II. β-metrics.

Definition 5. The metric d on X is called a β-metric if d is an α-metric having the property that if $L(a, b)$ is any line segment in X and $p \in X$, then $L(p, L(a, b))$ is convex.

Throughout the remainder of this paper we assume that d is a β-metric.

It might be well to indicate here the general thrust of the arguments to follow. Suppose, for the moment, that X is finite dimensional. We will construct in X a convex cell I_n with a nonvoid interior. We then collapse X into I_n by means of a 1-1 function which takes $\partial(X)$ onto bdry I_n. In order to establish the continuity of f we establish two important properties of $\partial(X)$. One is that any ray from a nonrim point of X to a rim point contains no other rim point. The second is that $\partial(X)$ is closed. The remaining arguments are directed toward constructing the cell mentioned above and toward establishing these two properties of $\partial(X)$.

Proposition 6. *If A is a convex subset of X and $p \in X - A$, then $L(p, A)$ is convex.*

It follows from the continuity of λ that if A is closed, $L(p, A)$ is closed.

Theorem 3. *Suppose p and q are distinct points of X, and $a \in X$ is not on the maximal line segment containing p and q. (It follows from Propositions* 3 *and* 6 *and Theorem* 2 *that $L(a, L(p, q))$ is a convex 2-cell.) Then each maximal line segment in the 2-cell $A = L(a, L(p, q))$ has its endpoints on the boundary of the cell.*

Proof. Suppose $L(s, w')$ is a maximal segment in A with $s \notin \text{bdry}(A)$. Choose a point $w \neq s$ with $w \in L(s, w')$ and $w \notin \text{bdry}(A)$. Then $L(w, s)$ is a maximal ray from w.

Define $F\colon \text{bdry}(A) \times [0, 1] \to A$ as follows: $F(x, t) = y$ where $y \in L(w, x)$ and $d(y, x)/d(w, x) = t$. Then F is a contraction of the boundary of A to w.

Now $F(x, t) \neq s$ for any (x, t); for otherwise $s \in L(x, w)$, and hence, since $x \neq s$, $L(w, s)$ is a proper segment of $L(w, x)$, contradicting the maximal property of $L(w, s)$.

Hence we have a contraction of bdry A to a point in $A - \{s\}$, which is impossible.

It follows from the above theorem that $\partial(A) \subset \text{bdry}\, A$. Theorem 5 asserts $\partial(A) = \text{bdry}\, A$.

Theorem 4. *Suppose $p \in X - \partial(X)$ and $q \in \partial(X)$. Then $L(p, q) \cap \partial(X) = \{q\}$.*

Proof. Suppose there exists an $s \in \partial(X) \cap L(p, q)$ and $s \neq q$. (See Figure 1.) Then $s \neq p$, for $p \notin \partial(X)$; and since $s \in \partial(X)$ there exists a point $t \in \partial(X)$ such that $L(s, t)$ is a maximal segment.

(1) If L is any segment which contains p and q, then $t \notin L$. For if $t \in L$, then since $s \in L$, $\{s, t\} \subset L \cap L(s, t)$, and therefore $L \cup L(s, t)$ is a segment. And since $L(s, t)$ is maximal, $L \cup L(s, t) = L(s, t)$. Therefore L is a subsegment of $L(s, t)$ containing s as an interior point. Since this is impossible, $t \notin L$.

Let $L(a, b)$ denote the maximal segment containing t and p; and let the order from a to b be $a \leq t < p < b$. (Again $t \neq p \neq b$, since $\{t, b\} \subset \partial(X)$.)

(2) Now $s \notin L(a, b)$. For if $s \in L(a, b)$, then $\{s, p\} \subset L(p, q) \cap L(a, b)$, and therefore $L(p, q) \cup L(a, b)$ is a segment. Since $L(a, b)$ is maximal, $L(p, q) \subset L(a, b)$. Hence $L(a, b)$ is a segment containing p, q, and t, which contradicts (1). So $s \notin L(a, b)$.

By (2), $q \notin L(a, b)$, for otherwise $s \in L(a, b)$. So by Proposition 3, $L(a, b) = F_q(L(a, b))$; and hence, by Proposition 2, $L(q, a) \cap L(q, p) = \{q\}$. Therefore we have

(3) $s \notin L(q, a)$.

Likewise,

(4) $s \notin L(q, b)$.

Now by Theorem 2, $L(q, L(a, b))$ is a convex 2-cell with $\text{bdry}\,(L(q, L(a, b))) = L(a, b) \cup L(q, a) \cup L(q, b)$. Also $L(s, t) \subset L(q, L(a, b))$, and by (2), (3), and (4), $s \notin \text{bdry}\,(L(q, L(a, b)))$. Hence, by Theorem 3, $L(s, t)$ is not maximal in $L(q, L(a, b))$, and therefore is not maximal in X. This contradiction proves the theorem.

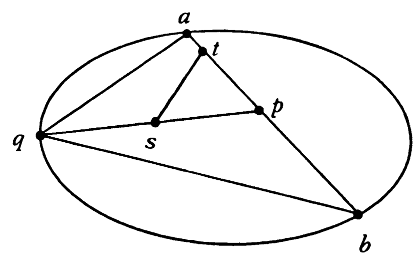

Figure 1

Theorem 5. *Suppose A is a nondegenerate closed convex subset of X, $A - \partial(A) \neq \emptyset$, $p \in X - A$, and $A = F_p(A)$. Then $\partial(L(p, A)) = L(p, \partial(A)) \cup A$.*

Before proving Theorem 5, if $p \neq a \in L(p, A)$ define the projection of a on A, denoted by $p(a)$, to be that unique point $a' \in A$ such that $a \in L(p, a')$. It is easily shown that the map p is "linear" in the sense that if $p \notin L(a, b) \subset L(p, A)$, then $p(L(a, b)) = L(p(a), p(b))$.

Proof of Theorem 5. First we show $\partial(L(p, A)) \subset L(p, \partial A) \cup A$ by showing that if $L(a, b)$ is a maximal segment in $L(p, A)$, then $\{a, b\} \subset L(p, \partial A) \cup A$.

We may assume $p \notin \{a, b\}$, for otherwise $\{a, b\} \subset \{p\} \cup A \subset L(p, \partial A) \cup A$. Let

$p(a) = a' \neq b' = p(b)$ and let $L(x, y)$ be the maximal segment in A containing $\{a', b'\}$. Since $L(a, b)$ is a maximal segment in the convex 2-cell $L(p, L(x, y))$, by Theorem 3, $\{a, b\} \subset L(x, y) \cup L(p, x) \cup L(p, y) \subset L(p, \partial A) \cup A$.

Now we show $L(p, \partial A) \cup A \subset \partial(L(p, A))$. Suppose $a \in L(p, \partial A) \cup A$. Since it easily follows that $L(p, x)$ is a maximal segment in $L(p, A)$ for each $x \in A$, we assume $a \notin A$, $a \neq p$. Let $p(a) = a' \in \partial A$ and select $x \in A - \partial A$. It suffices to show that $L(x, a)$ is a maximal ray from x in $L(p, A)$.

Suppose, on the contrary, that $L(x, s)$ is a maximal ray from x in $L(p, A)$ containing $L(x, a)$ with $a \neq s$. Then $s \in \partial(L(p, A))$; $s \neq p$ (otherwise $a' = x \in \partial A$); and $s \notin A$. Hence $p(s) = s' \in \partial A$. Also $s' \neq a'$, for otherwise $\{s, a\} \subset L(p, a') \cap L(x, s)$ and therefore $L(p, a') \cup L(x, s) = L(p, a')$. Then $x = a'$.

Now since $p(L(x, s)) = L(x, s')$, we have $a' \in L(x, s')$. This violates Theorem 4. Hence $L(x, a)$ is a maximal ray from x in $L(p, A)$.

Corollary. *Suppose* $\{p_i\}_{i=0}^n$ *are distinct points of* X. *Set* $A_0 = \{P_0\}$ *and for* $1 \leq i \leq n$ *set* $A_i = L(p_i, A_{i-1})$. *Suppose also* $F_{p_i}(A_{i-1}) = A_{i-1}$ *for* $1 \leq i \leq n$ *so that, by Theorem* 2, A_i *is an* i-*cell. Then* $\partial(A_i) = \text{bdry } A_i$.

Proof. Induction on i, together with the remark following Theorem 2.

Theorem 6. *Suppose* A *is a closed convex subset of* X, $p \in X - A$ *and the sets* $F_p(A)$ *and* $A - \partial A$ *have a point* q *in common. Then* $F_p(A) = A$.

Proof. Suppose $x \in A$, $x \neq q$. We show $x \in F_p(A)$. Let $L(s, t)$ be the maximal segment in X containing $\{x, q\}$. By Proposition 4, $L(s, t) \cap A = L(a, b)$ is a maximal segment in A. To be specific we take the order on $L(a, b)$ as $a < q < x \leq b$ (see Figure 2).

Since $q \in F_p(A)$, $p \notin L(s, t)$. Hence $S = L(p, L(a, x))$ is a convex 2-cell with bdry $S = L(p, a) \cup L(a, x) \cup L(p, x)$. Now if $x \notin F_p(A)$, there exists a point $w \in A \cap L(p, x)$ with $w \neq x$. Hence the arc $L(a, w)$ which lies, except for its endpoints, in the interior of S and separates p and q in S also lies in A. Therefore $L(p, q)$ contains a point of A other than q, which contradicts the fact that $q \in F_p(A)$.

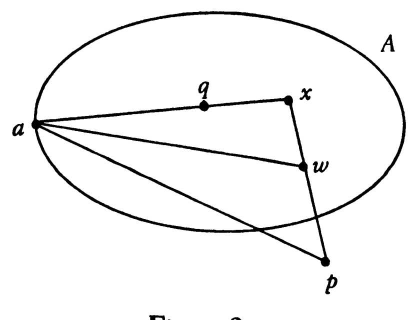

Figure 2

Proposition 7. *Suppose A is a closed convex subset of X, $\{q_1, q_2\} \subset A$, $q_1 \neq q_2$, $p \in X - A$ and is not on any line segment containing $\{q_1, q_2\}$, and $q_1 \notin F_p(A)$. If $a \in L(q_1, q_2)$, $a \notin \{q_1, q_2\}$, then $a \notin F_p(A)$.*

Proof. Very much like that for Theorem 6.

Theorem 7. *If A is a closed, convex subset of X, $\{p_1, p_2\} \subset X$, $p_1 \neq p_2$, with $L(p_1, p_2) \cap A = a \in A - \partial(A)$, then $B = L(p_1, A) \cup L(p_2, A)$ is closed, convex and $\partial B = L(p_1, \partial A) \cup L(p_2, \partial A)$.*

Proof. First if $p_1 \in A$, then $p_1 = a \in F_{p_2}(A)$. Hence, by Theorem 6, $A = F_{p_2}(A)$. Then by Theorem 5, $B = A \cup L(p_2, A) = L(p_2, A)$ is closed, convex and $\partial(B) = A \cup L(p_2, \partial(A)) = L(p_1, \partial(A)) \cup L(p_2, \partial(A))$. So we now suppose $\{p_1, p_2\} \subset X - A$, and hence $F_{p_1}(A) = A = F_{p_2}(A)$.

Lemma 1. $F_{p_2}(L(p_1, A)) = A$.

Proof. In view of Theorem 6 it is sufficient to show that if $q \in L(p_1, A) - A$, then $q \notin F_{p_2}(L(p_1, A))$.

It follows from Theorem 6 and the fact that $p_1 \notin F_{p_2}(L(p_1, A))$ that if $q \in L(p_1, A) - \partial L(p_1, A)$, then $q \notin F_{p_2}(L(p_1, A))$.

So suppose $p_1 \neq q \in \partial L(p_1, A) - A = (A \cup L(p_1, \partial(A)) - A = L(p_1 \partial A) - \partial A$. We may select a point $q_1 \in \partial A$, $q \neq q_1$, with $q \in L(p_1, q_1)$. Since $q_1 \neq a$, p_2 is not on any line segment containing p_1 and q_1. Hence by Proposition 7, $q \notin F_{p_2}(L(p_1, A))$. This proves Lemma 1.

From Lemma 1 it follows that $L(p_2, L(p_1, A)) = L(p_2, F_{p_2}(L(p_1, A))) \cup L(p_1, A) = L(p_2, A) \cup L(p_1, A) = B$ is closed and convex.

Lemma 2. *Each line segment in B which contains two points of A must lie in A.*

Proof. This follows easily from the fact that $F_{p_i}(A) = A$.

Lemma 3. *We now show that if $x \in A - \partial(A)$ then $L(p_i, x)$ is not maximal in B, $i = 1, 2$. (In particular $L(p_i, x) \subset L(p_i, x_0)$ with $x \neq x_0$, $x_0 \in B$.)*

Proof. If $x = a$, $L(p_i, x)$ is properly contained in $L(p_1, p_2)$; so suppose $x \neq a$.

Let $L(s, t)$ be the maximal segment in A containing a and x. Since $\{s, t\} \cap \{a, x\} = \varnothing$, let the order on $L(s, t)$ be $s < a < x < t$, and let $S = L(t, L(p_1, p_2))$. This is a convex 2-cell with $\partial S = \text{bdry } S = L(p_1, p_2) \cup L(t, p_1) \cup L(t, p_2)$.

Since $\{a, t\} \subset S$, $x \in S$. Hence $L(p_i, x) \subset S$. If $L(p_i, x)$ were maximal in B, $L(p_i, x)$ would be maximal in S. But since $x \notin \partial S$ we must have $L(p_i, x) \subset L(p_i, x_0)$ for some $x_0 \neq x$, $x_0 \in S \subset B$. This proves Lemma 3.

Lemma 4. *A separates B.*

Proof. This follows easily from the observation that $L(p_1, A) \cap L(p_2, A) = A$.

We are now ready to show $\partial B \subset L(p_1, \partial A) \cup L(p_2, \partial A)$. Suppose $L(s, t)$ is maximal in B; we show $\{s, t\} \subset L(p_1, \partial A) \cup L(p_2, \partial A)$.

In view of Lemmas 2, 4 and Theorem 5, we may assume that $L(s, t)$ contains exactly one point q of A.

Case I. $q \notin \{s, t\}$. If $\{s, t\} \subset L(p_1, A)$, then $L(s, t)$ is maximal in $L(p_1, A)$. Hence $\{s, t\} \subset \partial L(p_1, A) - A = (L(p_1, \partial A) \cup A) - A \subset L(p_1, \partial A)$. So suppose $s \in L(p_1, A)$ and $t \in L(p_2, A)$. Then $L(s, q) = L(s, t) \cap L(p_1, A)$ is maximal in $L(p_1, A)$, by Proposition 4, and $L(q, t) = L(s, t) \cap L(p_2, A)$ is maximal in $L(p_2, A)$. Hence $s \in \partial L(p_1, A) - A \subset L(p_1, \partial A)$ and $t \in \partial L(p_2, A) - A \subset L(p_2, \partial A)$.

Case II. Suppose $q = t$, and $s \in L(p_1, A) - A$. Then $L(s, t)$ is maximal in $L(p_1, A)$ and hence $s \in L(p_1, \partial A)$. We now show $q \in \partial A \subset L(p_1, \partial A)$. Suppose $q \notin \partial A$. Let $s' \in \partial A$ such that $s \in L(p_1, s')$. We have $s' \neq q$. Let $L(x, y)$ be the maximal segment in A containing $\{s', q\}$, and let the order be $x \leq s' < q < y$. Choose r so that $q < r < y$. Since $q \notin \partial A$ and $y \in \partial A$ by Theorem 4, $r \notin \partial A$. Hence, by Lemma 3, there exists an $r_0 \in B$ such that $L(p_1, r) \subset L(p_1, r_0)$ and $r \neq r_0$.

Now consider $S = L(x, L(p_1, r_0))$. Since $\{x, r\} \subset S$, $q \in S$; and since $L(p_1, s') \subset S$, $s \in S$. Hence $L(s, q) \subset S$, and hence is maximal in S. So $q \in \partial S$.

Now $\partial S = \text{bdry } S = L(p_1, r_0) \cup L(x, p_1) \cup L(x, r_0)$. Since $r \in L(p_1, r_0)$, $q \notin L(p_1, r_0)$ (otherwise not both r and q are in $F_{p_1}(A) = A$). Likewise $q \notin L(x, p_1)$.

Since $r \neq r_0$, and since x is not on any line segment containing p_1 and r_0 (otherwise not both x and r could be in $F_{p_1}(A)$), $L(x, r) \cap L(x, r_0) = \{x\}$. So since $q \in L(x, r)$, $q \notin L(x, r_0)$. Hence $q \notin \partial S$. This contradiction establishes Case II. Therefore $\partial B \subset L(p_1, \partial A) \cup L(p_2, \partial A)$.

To see that $L(p_1, \partial A) \cup L(p_2, \partial A) \subset \partial B$, suppose $x \in L(p_1, \partial A)$. We show $x \in \partial B$. Since $L(p_1, p_2)$ is maximal in B, suppose $x \neq p_1$.

If $x \in \partial A$, it follows from Lemma 2 that $x \in \partial B$. If $x \notin \partial A$, then as, in the proof of Theorem 5, $L(a, x)$ is a maximal ray from a in $L(p_1, A)$. Hence, by Lemmas 2 and 4, $L(a, x)$ is a maximal ray from a in B. Therefore $x \in \partial B$.

Theorem 8. *If X is finite dimensional, then $\partial(X)$ is closed.*

Proof. Suppose $x_0 \in X - \partial(X)$. Let $l_1 = L(p_0, p_1)$ be any maximal line segment containing x_0. Since $x_0 \notin \partial(X)$, $p_0 \neq x_0 \neq p_1$.

Either l_1 contains a neighborhood of x_0, or it does not. Suppose it does not. Choose $\epsilon > 0$ so that $S_\epsilon(x_0)$ misses $\{p_0, p_1\}$. Select a point $x \in S_\epsilon(x_0)$ with $x \notin l_1$. Since $L(x, x_0) \subset S_\epsilon(x_0)$, $L(x, x_0) \cap \partial l_1 = L(x, x_0) \cap \{p_0, p_1\} = \varnothing$. Hence by Proposition 5, $x_0 \in F_x(l_1)$ and therefore $L(x, x_0) \cap l_1 = \{x_0\}$.

Let $L(p_2, p_3)$ be the maximal segment in X containing $\{x, x_0\}$. Then $L(p_2, p_3) \cap I_1 = \{x_0\}$ and $p_2 \neq x_0 \neq p_3$. Set $I_2 = L(p_2, I_1) \cup L(p_3, I_1)$. By Theorem 7, I_2 is closed, convex and $\partial I_2 = L(p_2, \partial I_1) \cup L(p_3, \partial I_1) = L(p_2, \{p_0, p_1\}) \cup L(p_3, \{p_0, p_1\})$. Also I_2 is 2-dimensional, since it is the sum of two 2-cells.

If I_2 contains no neighborhood of x_0, then choose $\epsilon > 0$ so that $S_\epsilon(x_0)$ misses ∂I_2 and select a point $y \in S_\epsilon(x_0) - I_2$. Again since $L(y, x_0) \subset S_\epsilon(x_0)$, $L(y, x_0)$ misses ∂I_2 and thus $x_0 \in F_y(I_2)$; hence by Theorem 6, $I_2 = F_y(I_2)$. Let $L(p_4, p_5)$ be the maximal segment in X containing $\{y, x_0\}$. Set $I_3 = L(p_4, I_2) \cup L(p_5, I_2)$. Then I_3 is closed, convex and $\partial I_3 = L(p_4, \partial I_2) \cup L(p_5, \partial I_2)$ and since $p_4 \neq x_0 \neq p_5$, $x_0 \notin \partial I_3$. Also I_3 is 3-dimensional, since it is the sum of four 3-cells.

If I_3 contains no neighborhood of x_0, we repeat the process until eventually, since X is finite dimensional, we arrive at a closed convex set I_n with $x_0 \in I_n - \partial I_n$ and $\epsilon > 0$ such that $S_\epsilon(x_0) \subset I_n - \partial I_n \subset X - \partial X$.

Theorem 9. *If X is finite dimensional, then X is homeomorphic to an n-cell for some $n \geq 1$.*

Proof. The proof is similar in many respect to that of Theorem 8.

Let $p_0 \neq p_1$ be any two points of X. Let a_1 denote the midpoint of $L(p_0, p_1) = I_1$. Either I_1 contains a neighborhood of a_1 or it does not.

If it does not, choose $\epsilon > 0$ so that $S_\epsilon(a_1) \cap \partial I_1 = S_\epsilon(a_1) \cap \{p_0, p_1\} = \emptyset$. Then select a point $p_3 \in S_\epsilon(a_1) - I_1$. As in the proof of Theorem 8, $L(p_3, a_1) \subset S_\epsilon(a_1)$ and hence $F_{p_3}(I_1) = I_1$. Consequently $L(p_3, I_1) = I_2$ is a convex 2-cell. Let a_2 denote the midpoint of $L(p_3, a_1)$, and note that $a_2 \notin \partial I_2 = I_1 \cup L(p_3, \partial I_1)$.

Again, either I_2 contains a neighborhood of a_2, or it does not. If if does not, we proceed as above until we obtain a convex n-cell, I_n, and a point $a_n \in I_n - \partial I_n = I_n - \text{bdry } I_n$ such that I_n contains a neighborhood of a_n. And since $a_n \notin \partial I_n$, $a_n \notin \partial X$.

Now if $x \in \partial X$, by Theorem 4, $L(a_n, x)$ is a maximal ray from a_n in X. Hence if follows from Proposition 4 that $L(a_n, x) \cap I_n$ is a maximal ray from a_n in I_n. Hence $L(a_n, x) \cap I_n = L(a_n, y)$ where $L(a_n, y) \cap \partial I_n = \{y\}$.

Conversely if $y_0 \in \partial I_n$, then $L(a_n, y_0)$ is a maximal ray in I_n with $L(a_n, y_0) \cap \partial I_n = \{y_0\}$ and if $L(a_n, x_0)$ is the maximal ray in X containing $L(a_n, y_0)$, then $L(a_n, x_0) \cap \partial X = \{x_0\}$.

Now we define $f\colon X \to I_n$ as follows: $f(a_n) = a_n$; and if $x \neq a_n$, let $L(a_n, x_0)$ be the maximal ray in X from a_n through x and $L(a_n, y_0) = I_n \cap L(a_n, x_0)$ where $x_0 \in \partial X$ and $y_0 \in \partial I_n$, then $f(x)$ is that point on $L(a_n, y_0)$ which is the image of x under the mapping which takes $L(a_n, x_0)$ linearly on $L(a_n, y_0)$ with $f(a_n) = a_n$ and $f(x_0) = y_0$.

Explicitly, if $\lambda(a_n, x_0, r) = x$, $0 \le r \le 1$, then $f(x) = \lambda(a_n, y_0, r)$.

Now f is a 1-1 function of X onto I_n taking $\partial(X)$ onto ∂I_n.

In view of the continuity of λ and the fact that $\partial I_n = \text{bdry } I_n$ is closed, the continuity of f results from the following:

Lemma. *If* $\{p_i\}_1^\infty \to p_0$ *and* $p_i \neq a_n$, $i \ge 0$, *and if* $L(a_n, x_i)$ *is the maximal ray in* X *from* a_n *through* p_i, $i \ge 0$, *then* $\{x_i\}_1^\infty \to x_0$.

Proof. Let $\{x_{n_i}\}_1^\infty \to x_0'$ be any convergent subsequence of $\{x_i\}_1^\infty$. We show $x_0' = x_0$. Since $\{L(a_n, x_{n_i})\} \to L(a_n, x_0')$, $p_0 \in L(a_n, x_0')$. Hence $L(a_n, x_0') \subset L(a_n, x_0)$. Since $\{x_i\}_0^\infty \subset \partial X$ and ∂X is closed, $x_0' \in \partial X$. Hence, again by Theorem 4, $x_0' = x_0$.

III. **Questions.** It is an open question, asked by Borsuk and Bing, whether or not each finite dimensional compact metric space X which admits an α-metric is homeomorphic to a cell. If $\dim X = n$, then Lelek and Nitka [7] supply an affirmative answer if $n \le 2$. Rolfsen [3] proves that X is a cell if $n = 3$; and he also shows that if X is a manifold and $n > 5$, then X is a cell.

Question 1. If X is a finite dimensional compact space with α-metric d, can one use d to define a β-metic on X?

Question 2. Is each compact infinite dimensional space X with an α-metric homeomorphic to the Hilbert cube?

Question 3. Same as above but assuming the existence of a β-metric on X.

Question 4. Is there a β-metric on the Hilbert cube such that each point is a rim point?

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